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# Tight Approximation Bounds for Vertex Cover on Dense $k$-Partite Hypergraphs

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**Abstract**

We establish almost tight upper and lower approximation bounds for the Vertex Cover problem on dense $k$-partite hypergraphs.

## 1 Introduction

A hypergraph $H = (V, E)$ consists of a vertex set $V$ and a collection of hyperedges $E$ where a hyperedge is a subset of $V$. $H$ is called $k$-uniform if every edge in $E$ contains exactly $k$ vertices. A subset $C$ of $V$ is a vertex cover of $H$ if every edge $e \in E$ contains at least a vertex of $C$.

The *Vertex Cover* problem in a $k$-uniform hypergraph $H$ is the problem of computing a minimum cardinality vertex cover in $H$. It is well known that the problem is $NP$-hard even for $k = 2$ (cf. [13]). On the other hand, the simple greedy heuristic which chooses a maximal set of nonintersecting edges, and then outputs all vertices in those edges, gives a $k$-approximation algorithm for the Vertex Cover problem restricted to $k$-uniform hypergraphs. The best known approximation algorithm achieves a slightly better approximation ratio of $(1 - o(1))k$ and is due to Halperin [11].

On the intractability side, Trevisan [22] provided one of the first inapproximability results for the $k$-uniform vertex cover problem and obtained a inapproximability factor of $k^{\frac{1}{19}}$ assuming $P \neq NP$. In 2002, Holmerin [11] improved the factor to $k^{1-\epsilon}$. Dinur et al. [7, 8] gave consecutively two lower bounds, first $(k - 3 - \epsilon)$ and later on $(k - 1 - \epsilon)$. Moreover, assuming Khot's Unique Games Conjecture (UGC) [17], Khot and Regev [18] proved an inapproximability factor of $k - \epsilon$ for the Vertex Cover problem on $k$-uniform hypergraphs. Therefore, it implies that the currently achieved ratios are the best possible.

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The Vertex Cover problem restricted to $k$-partite $k$-uniform hypergraphs, when the underlying partition is given, was studied by Lovász [20] who achieved a $\frac{k}{2}$-approximation. This approximation upper bound is obtained by rounding the natural LP relaxation of the problem. The above bound on the integrality gap was shown to be tight in [1]. As for the lower bounds, Guruswami and Saket [10] proved that it is NP-hard to approximate the Vertex Cover problem on $k$-partite $k$-uniform hypergraphs to within a factor of $\frac{k}{4} - \epsilon$ for $k \geq 5$. Assuming the Unique Games Conjecture, they also provided an inapproximability factor of $\frac{k}{2} - \epsilon$ for $k \geq 3$. More recently, Sachdeva and Saket [21] claimed a nearly optimal $NP$-hardness factor.

To gain better insights on lower bounds, dense instances of many optimization problems has been intensively studied [2, 15, 16, 14]. The Vertex Cover problem has been investigated in the case of dense graphs, where the number of edges is within a constant factor of $n^2$, by Karpinski and Zelikovsky [16], Eremeev [9], Clementi and Trevisan [6], later by Bar-Yehuda and Kehat [4] as well as Imamura and Iwama [12].

The Vertex Cover problem restricted to dense balanced $k$-partite $k$-uniform hypergraphs was introduced and studied in [5], where it was proved that this restricted version of the problem admits an approximation ratio better than $\frac{k}{2}$ if the given hypergraph is dense enough.

In this paper, we give a new approximation algorithm for the Vertex Cover problem restricted to dense $k$-partite $k$-uniform hypergraphs and prove that the achieved approximation ratio is almost tight assuming the Unique Games Conjecture.

# 2 Definitions and Notations

Given a natural number $i \in \mathbb{N}$, we introduce for notational simplicity the set $[i] = \{1, .., i\}$ and set $[0] = \emptyset$. Let $S$ be a finite set with cardinality $s$ and $k \in [s]$. We will use the abbreviation $\binom{S}{k} = \{S' \subseteq S \mid |S'| = k\}$.

A *$k$-uniform hypergraph* $H = (V(H), E(H))$ consists of a set of vertices $V$ and a collection $E \subseteq \binom{V}{k}$ of edges. For a $k$-uniform hypergraph $H$ and a vertex $v \in V(H)$, we define the *neighborhood* $N_H(v)$ of $v$ by $\left(\bigcup_{e \in \{e \in E \mid v \in e\}} e\right) \setminus \{v\}$ and the *degree* $d_H(v)$ of $v$ to be $|\{e \in E \mid v \in e\}|$. We extend this notion to subsets of $V(H)$, where $S \subseteq V(H)$ obtains the degree $d_H(S)$ by $|\{e \in E \mid S \subseteq e\}|$.

A *$k$-partite* $k$-uniform hypergraph $H = (V_1, .., V_k, E(H))$ is a $k$-uniform hypergraph such that $V$ is a disjoint union of $V_1, .., V_k$ with $|V_i \cap e| = 1$ for every $e \in E$ and $i \in [k]$. In the remainder, we assume that $|V_i| \geq |V_{i+1}|$ for all $i \in [k-1]$ and $k = O(1)$.

A *balanced* $k$-partite $k$-uniform hypergraph $H = (V_1, .., V_k, E(H))$ is a $k$-partite $k$-uniform hypergraph with $|V_i| = \frac{|V|}{k}$ for all $i \in [k]$. We set $n = |V|$ and $m = |E|$ as usual.

For a $k$-partite $k$-uniform hypergraph $H = (V_1, .., V_k, E(H))$ and $v \in V_k$, we introduce the *$v$-induced hypergraph* $H(v)$, where the edge set of $H(v)$ is defined by

$\{e \setminus \{v\} \mid v \in e \in E(H)\}$ and the vertex set of $H(v)$ is partitioned into $V_i \cap N_H(v)$ with $i \in [k-1]$.

A *vertex cover* of a $k$-uniform hypergraph $H = (V(H), E(H))$ is a subset $C$ of $V(H)$ with the property that $e \cap C \neq \emptyset$ holds for all $e \in E(H)$. The Vertex Cover problem consists of finding a vertex cover of minimum size in a given $k$-uniform hypergraph. The Vertex Cover problem in $k$-partite $k$-uniform hypergraphs is the restricted problem, where a $k$-partite $k$-uniform hypergraph and its vertex partition is given as a part of the input.

We define a $k$-partite $k$-uniform hypergraph $H = (V_1, .., V_k, E(H))$ as $\epsilon$-*dense* for an $\epsilon \in [0, 1]$ if the following condition holds:

$$|E(H)| \quad \geq \quad \epsilon \prod_{i \in [k]} |V_i|$$

For $\ell \in [k-1]$, we introduce the notion of $\ell$-wise $\epsilon$-dense $k$-partite $k$-uniform hypergraphs. Given a $k$-partite $k$-uniform hypergraph $H$, if there exists an $I \in \binom{[k]}{\ell}$ and an $\epsilon \in [0, 1]$ such that for all $S$ with the property $|V_i \cap S| = 1$ for all $i \in I$ the condition

$$d_H(S) \quad \geq \quad \epsilon \prod_{i \in [k] \setminus I} |V_i|$$

holds, we define $H$ to be $\ell$-wise $\epsilon$-dense.

# 3 Our Results

In this paper, we give an improved approximation upper bound for the Vertex Cover problem restricted to $\epsilon$-dense $k$-partite $k$-uniform hypergraphs. The approximation algorithm in [5] yields an approximation ratio of

$$\frac{k}{k - (k-2)(1-\epsilon)^{\frac{1}{k-\ell}}}$$

for $\ell$-wise $\epsilon$-dense balanced $k$-partite $k$-uniform hypergraphs. Here, we design an algorithm with an approximation factor of

$$\frac{k}{2 + (k-2)\epsilon}$$

for the $\epsilon$-dense case which also improves on the $\ell$-wise $\epsilon$-dense balanced case for all $\ell \in [k-2]$ and matches their bound when $\ell = k-1$. A further advantage of this algorithm is that it applies to a larger class of hypergraphs since the considered hypergraph is not necessarily required to be balanced.
As a byproduct, we obtain a constructive proof that a vertex cover of an $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, .., V_k, E(H))$ is bounded from below by $\epsilon|V_k|$, which is shown to be sharp by constructing a family of tight examples.

On the other hand, we provide inapproximability results for the Vertex Cover problem restricted to $\ell$-wise $\epsilon$-dense balanced $k$-partite $k$-uniform hypergraphs under the Unique Games Conjecture. We also prove that this reduction yields a

matching lower bound if we use a conjecture on the Unique Games hardness of the Vertex Cover problem restricted to balanced $k$-partite $k$-uniform hypergraphs. This means that further restrictions such as $\ell$-wise density cannot lead to improved approximation ratios and our proposed approximation algorithm is best possible assuming this conjecture. In addition, we are able to prove an inapproximability factor under $P \neq NP$.

# 4 Approximation Algorithm

In this section, we give a polynomial time approximation algorithm with improved approximation factor for the Vertex Cover problem restricted to $\epsilon$-dense $k$-partite $k$-uniform hypergraphs.

We state now our main result.

**Theorem 1.** *There exists a polynomial time approximation algorithm with approximation ratio*

$$\frac{k}{2 + (k-2)\epsilon}$$

*for the Vertex Cover problem in $\epsilon$-dense $k$-partite $k$-uniform hypergraphs.*

A crucial ingredient of the proof of Theorem 1 is Lemma 1, in which we show that we can extract efficiently a large part of an optimal vertex cover of a given $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, .., V_k, E(H))$. More precisely, we obtain in this way a constructive proof that the size of a vertex cover of $H$ is bounded from below by $\epsilon|V_k|$. The procedure for the extraction of a part of an optimal vertex cover is given in Figure 1.

We now formulate Lemma 1:

**Lemma 1.** *Let $H = (V_1, .., V_k, E(H))$ be an $\epsilon$-dense $k$-partite $k$-uniform hypergraph with $k \geq 1$. Then, the procedure $Extract(\cdot)$ computes in polynomial time a collection $R$ of subsets of $V(H)$ such that the size of $R$ is polynomial in $|V(H)|$ and $R$ contains a set $S$, which is a subset of an optimal vertex cover of $H$ and its cardinality is at least $\epsilon|V_k|$.*

As a consequence, we obtain directly:

**Corollary 1.** *Given an $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, .., V_k, E(H))$ with $k \geq 1$, the cardinality of an optimal vertex cover of $H$ is bounded from below by $\epsilon|V_k|$.*

Before we prove Lemma 1, we describe the main idea of the proof. Let $OPT$ denote an optimal vertex cover of $H$. The procedure $Extract(\cdot)$ tests for the set $R = \{v_1, .., v_p\}$ of the $p$ heaviest vertices of $V_k$, if $\{v_1, .., v_{u-1}\} \subseteq OPT$ and $v_u \notin OPT$ for every $u \in [p]$. Clearly, either $R \subseteq OPT$ or there exists a $v_u$ such that $v_u \notin OPT$. If the procedure already possesses a part of $OPT$ denoted by $R_u$, then,

**Procedure** $Extract(\cdot)$

Input: $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, .., V_k, E)$ with $k \geq 1$

1. IF $k = 1$ THEN
   (a) RETURN $\{\bigcup_{e \in E} e\}$
2. ELSE:
   (a) Let $(v_1, .., v_p)$ be the vector consisting of the first $p = \left\lceil \frac{|E|}{\prod_{l \in [k-1]} |V_l|} \right\rceil$ heaviest vertices of $V_k$ with $d_H(v_i) \geq d_H(v_{i+1})$
   (b) $R = \{\{v_1, .., v_p\}\}$
   (c) FOR $i = 1, .., p$ DO:
      i. $R_i = \{v_k \mid k \in [i-1]\}$
      ii. Invoke $Extract(H(v_i))$ with output $O$
      iii. $R = R \cup \{R_i \cup S \mid S \in O\}$
3. RETURN $R$

Figure 1: Procedure $Extract$

$Extract(\cdot)$ tries to obtain a large part of an optimal vertex cover of the $v_u$-induced hypergraph $H(v_u)$. Hence, we have to show that $H(v_u)$ must still be dense enough. We now give the proof of Lemma 1.

*Proof.* The proof of Lemma 1 will be split in several parts. In particular, we show that given an $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, .., V_k, E(H))$, the procedure $Extract(\cdot)$ and its output $R$ possess the following properties:

1. $Extract(\cdot)$ constructs $R$ in polynomial time and the cardinality of $R$ is $O(n^k)$.
2. There is a $S \in R$ such that $S$ is a subset of an optimal vertex cover of $H$.
3. For every $S \in R$, the cardinality of $S$ is at least $|S| \geq \epsilon |V_k|$.

(1.) Clearly, $R$ is upper bounded by $|V_1|^k = O(n^k)$ and therefore, the running time of $Extract(\cdot)$ is $O(n^k)$.
(2.) and (3.) We prove the remaining properties by induction. If we have $k = 1$, the set $\bigcup_{e \in E(H)} e$ is by definition an optimal vertex cover of $H = (V_1, E(H))$. Since $H$ is $\epsilon$-dense, the cardinality of $|E(H)|$ is lower bounded by $\epsilon |V_1|$.
We assume that $k > 1$. Let $H = (V_1, .., V_k, E(H))$ be an $\epsilon$-dense $k$-partite $k$-uniform hypergraph and $OPT \subseteq V(H)$ an optimal vertex cover of $H$. Let $(v_1, .., v_p)$ be the vector consisting of the first $p = \left\lceil \frac{|E(H)|}{\prod_{l \in [k-1]} |V_l|} \right\rceil$ heaviest vertices of $V_k$ with $d_H(v_i) \geq d_H(v_{i+1})$. If $\{v_1, .., v_p\}$ is contained in $OPT$, we have constructed a subset of an

optimal vertex cover with cardinality

$$p = \left\lceil \frac{|E(H)|}{\prod\limits_{l\in[k-1]} |V_l|} \right\rceil \quad \geq \quad \frac{\epsilon \prod\limits_{l\in[k]} |V_l|}{\prod\limits_{l\in[k-1]} |V_l|} \quad \geq \quad \epsilon|V_k|.$$

Otherwise, there is an $u \in [p]$ such that $R_u \subseteq OPT$ and $v_u \notin OPT$. But this means that an optimal vertex cover of $H$ contains an optimal vertex cover of the $v_u$-induced $(k-1)$-partite $(k-1)$-uniform hypergraph $H(v_u)$ in order to cover the edges $e \in \{e \in E \mid v_u \in e\}$. The situation is depicted in Figure 2.

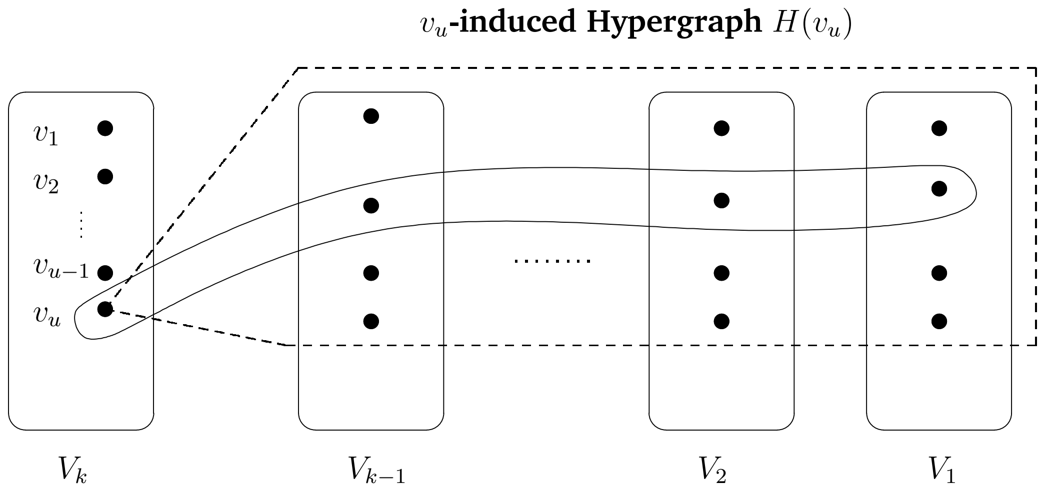


Figure 2: The $v_u$-induced $(k-1)$-partite $(k-1)$-uniform hypergraph $H(v_u)$

By our induction hypothesis, $Extract(H(v_u))$ contains a set $S_u$ which is a subset of a minimum vertex cover of $H(v_u)$ and of $OPT$. The only claim, which remains to be proven, is that the cardinality of $S_u$ is large enough. More precisely, we show that $|S_u|$ can be lower bounded by $\epsilon|V_k| - |R_u|$. Therefore, we need to analyze the density of the $v_u$-induced hypergraph $H(v_u)$. The edge set of $H(v_u)$ is given by $\{e \setminus \{v_u\} \mid v_u \in e \in E\}$. Thus, we have to obtain a lower bound on the degree of $v_u$. Since $|\{e \in E \mid e \cap R_u \neq \emptyset\}|$ is upper bounded by $|R_u| \prod_{l\in[k-1]} |V_l|$, the vertices in $V_k \setminus R_u$ possess the average degree of at least

$$\frac{\sum\limits_{v\in V_k\setminus R_u} deg_H(v)}{|V_k \setminus R_u|} \geq \frac{\epsilon \prod\limits_{l\in[k]} |V_l| - |\{e \in E \mid e \cap R_u \neq \emptyset\}|}{|V_k \setminus R_u|} \tag{1}$$

$$\geq \frac{\epsilon \prod\limits_{l\in[k]} |V_l| - |R_u| \prod\limits_{l\in[k-1]} |V_l|}{|V_k \setminus R_u|} \tag{2}$$

$$\geq \frac{(\epsilon|V_k| - |R_u|) \prod\limits_{l\in[k-1]} |V_l|}{|V_k \setminus R_u|} \tag{3}$$

Since the heaviest vertex in $V_k \setminus R_u$ must have a degree of at least $\frac{(\epsilon|V_k|-|R_u|)\prod_{l\in[k-1]}|V_l|}{|V_k\setminus R_u|}$, we deduce that the edge set of $H(v_u)$ denoted by $E_u$ can be lower bounded by

$$|E_u| \quad \geq \quad \frac{(\epsilon|V_k| - |R_u|) \prod\limits_{l\in[k-1]} |V_l|}{|V_k \setminus R_u|}$$

Let $H(v_u)$ be defined by $(V_1^u, .., V_{k-1}^u, E_u)$ with $|V_i^u| \leq |V_i|$ for all $i \in [k-1]$. By our induction hypothesis, the size of every set contained in $Extract(\cdot)$ is at least

$$\frac{|E_u|}{\prod\limits_{l\in[k-1]} |V_l^u|}|V_{k-1}| \quad \geq \quad \frac{(\epsilon|V_k| - |R_u|) \prod\limits_{l\in[k-1]} |V_l|}{|V_k \setminus R_u| \prod\limits_{l\in[k-1]} |V_l^u|}|V_{k-1}| \tag{4}$$

$$\geq \quad \frac{(\epsilon|V_k| - |R_u|) \prod\limits_{l\in[k-1]} |V_l|}{|V_k \setminus R_u| \prod\limits_{l\in[k-1]} |V_l|}|V_{k-1}| \tag{5}$$

$$\geq \quad \frac{(\epsilon|V_k| - |R_u|)}{|V_k \setminus R_u|}|V_k| \tag{6}$$

$$\geq \quad \frac{(\epsilon|V_k| - |R_u|)}{|V_k|}|V_k| \; = \; \epsilon|V_k| - |R_u| \tag{7}$$

In $(4)$, we used the fact that $|V_i^u| \leq |V_i|$ for all $i \in [k-1]$. Whereas in $(5)$, we used our assumption $|V_k| \leq |V_{k-1}|$. All in all, we obtain

$$|R_u \cup S_u| \quad \geq \quad |R_u| + (|\epsilon|V_k| - |R_u|) \; = \; \epsilon|V_k|. \tag{8}$$

Clearly, this argumentation on the size of $R_u \cup S_u$ holds for every $u \in [p]$ and the proof of Lemma 1 follows. ☐

Before we state our approximation algorithm and prove Theorem 1, we show that the bound in Lemma 1 is tight. In particular, we define a family of $\epsilon$-dense $k$-partite $k$-uniform hypergraphs $H(k, l, \epsilon) = (V_1, .., V_k, E(H_l))$ with $|V_i| = \frac{|V|}{k}$ for all $i \in [k]$, $k \geq 1$, $\epsilon \in \{\frac{u}{l} \mid u \in [l]\}$ and $l \geq 1$ such that $Extract(\cdot)$ returns a subset of an optimal vertex cover with cardinality of exactly $\epsilon|V_k|$.

**Lemma 2.** *The bound of Lemma 1 is tight.*

*Proof.* Let us define $H(k, p, \epsilon) = (V_1, .., V_k, E)$. For a fixed $p \geq 1$ and $k \geq 1$, every partition $V_i$ with $i \in [k]$ consists of a set of $l$ vertices. Let us fix a $\epsilon = \frac{u}{l}$ with $u \in [l]$. Then, $H(k, l, \epsilon)$ contains the set $V_k^u \subseteq V_k$ of $u$ vertices such that $E = \{\{v_1, v_2, .., v_k\} \mid v_1 \in V_k^u, v_2 \in V_2, .., v_k \in V_k\}$. An example of such a hypergraph is depicted in Figure 3.

Notice that $H(k, l, \epsilon) = (V_1, .., V_k, E)$ is $\epsilon$-dense, since

$$\frac{|E|}{\prod\limits_{j\in[k]} |V_j|} = \frac{|V_k^u|}{|V_k|} = \frac{u}{l} = \epsilon.$$

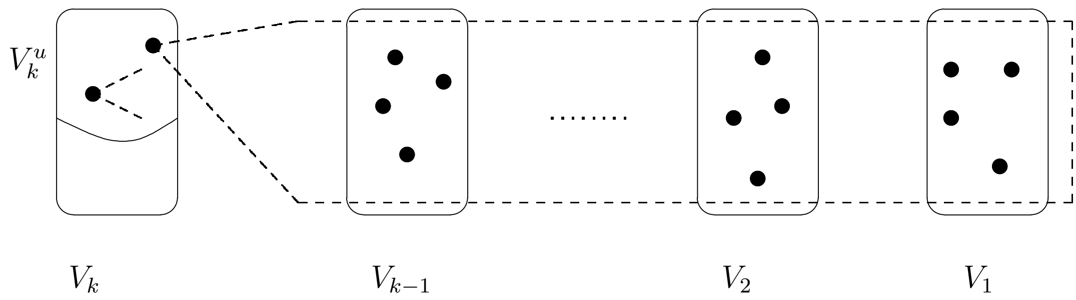


Figure 3: An example of a hypergraph $H(k, l, \epsilon)$

The procedure $Extract(\cdot)$ returns a set $R$, in which $V_k^u$ is contained, since $V_k^u$ is the set of the $p$ heaviest vertices of $V_k$. Hence, we obtain $|V_k^u| = \frac{|V_k^u|}{|V_k|}|V_k| = \epsilon|V_k|$. On the other hand, the remaining hypergraph $H' = (V_1, .., V_k \setminus V_k^u, E(H'))$ with edge set $E(H') = \{e \in E \mid e \cap V_k^u = \emptyset\}$ is already covered, since $E(H')$ is by definition of $H(k, p, \epsilon)$ the empty set. Therefore, $V_k^u$ is a vertex cover of $H(k, p, \epsilon)$ and since, according to Corollary 1, every vertex cover is bounded from below by $\epsilon|V_k|$, $V_k^u$ must be an optimal vertex cover. $\square$

Next, we state our approximation algorithm for the Vertex Cover problem in $\epsilon$-dense $k$-partite $k$-uniform hypergraphs defined in Figure 4. The approximation algorithm combines the procedure $Extract(\cdot)$ to generate a large enough subset of an optimal vertex cover together with the $\frac{k}{2}$-approximation algorithm due to Lovász [20] applied to the remaining instance.

**Algorithm** $Approx(\cdot)$

Input: $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, .., V_k, E)$ with $k \geq 3$

1. $T = \{V_k\}$
2. invoke procedure $Extract(H)$ with output $R$
3. for all $S \in R$ do :
   (a) $H_S = (V(H) \setminus S, \{e \in E(H) \mid e \cap S = \emptyset\})$
   (b) obtain a $(\frac{k}{2})$-approximate solution $S_k$ for $H_S$
   (c) $T = T \cup \{S_k \cup S\}$
4. Return the smallest set in $T$

Figure 4: Algorithm $Approx(\cdot)$

We now prove Theorem 1.

*Proof.* Let $H = (V_1, .., V_k, E)$ be an $\epsilon$-dense $k$-partite $k$-uniform hypergraph. From Lemma 1, we know that the procedure $Extract(\cdot)$ returns in polynomial time a collection $C$ of subsets of $V(H)$ such that there is a set $S$ in $C$, which is contained in an optimal vertex cover of $H$. Moreover, we know that the size of $S$ is lower bounded by $\epsilon|V_k|$.
Next, we analyze the approximation ratio of our approximation algorithm $Approx(\cdot)$. Clearly, the size of an optimal vertex cover of $H$ is upper bounded by $|V_k|$. Let us denote by $OPT'$ the size of an optimal vertex cover of the remaining hypergraph $H'$ defined by removing all edges $e$ of $H$ with $e \cap S \neq \emptyset$. Furthermore, let $S'$ be the solution of the $\frac{k}{2}$-approximation algorithm applied to $H'$. The approximation ratio of $Approx(\cdot)$ is bounded by

$$\frac{|S| + |S'|}{|S| + |OPT'|} \leq \frac{|S| + \frac{k}{2}|OPT'|}{|S| + |OPT'|} \leq \frac{k}{\frac{k|S|+k|OPT'|}{|S|+\frac{k}{2}|OPT'|}} \tag{9}$$

$$\leq \frac{k}{\frac{2|S|+(k-2)|S|+k|OPT'|}{|S|+\frac{k}{2}|OPT'|}} \tag{10}$$

$$\leq \frac{k}{2 + (k-2)\frac{|S|}{|S|+\frac{k}{2}|OPT'|}} \tag{11}$$

$$\leq \frac{k}{2 + (k-2)\frac{|S|}{|V_k|}} \tag{12}$$

$$\leq \frac{k}{2 + (k-2)\frac{\epsilon|V_k|}{|V_k|}} \tag{13}$$

$$\leq \frac{k}{2 + (k-2)\epsilon} \tag{14}$$

In $(11)$, we used the fact that the size of the output of $Approx(\cdot)$ is upper bounded by $|V_k|$. Therefore, we have $|S| + \frac{k}{2}|OPT'| \leq |V_k|$. In $(12)$, we know from Lemma 1 that $|S| \geq \epsilon|V_k|$. □

# 5 Inapproximability Results

In this section, we prove hardness results for the Vertex Cover problem restricted to $\ell$-wise $\epsilon$-dense balanced $k$-uniform $k$-partite hypergraphs under the Unique Games Conjecture [17] as well as under the assumption $P \neq NP$.

## 5.1 UGC-Hardness

The Unique Games-hardness result of [10] was obtained by applying the result of Kumar et al. [19], with a modification to the LP integrality gap due to Ahorani et al. [1]. More precisely, they proved the following inapproximability result:

**Theorem 2.** *[10] For every $\delta > 0$ and $k \geq 3$, there exist a $n_\delta$ such that given $H = (V_1, .., V_k, E(H))$ as an instance of the Vertex Cover problem in balanced $k$-partite $k$-uniform hypergraphs with $|V(H)| \geq n_\delta$, the following is UGC-hard to decide:*

- *The size of a vertex cover of $H$ is at least $|V| \left(\frac{1}{2(k-1)} - \delta\right)$.*
- *The size of an optimal vertex cover of $H$ is at most $|V| \left(\frac{1}{k(k-1)} + \delta\right)$.*

As the starting point of our reduction, we use Theorem 2 and prove the following:

**Theorem 3.** *For every $\delta > 0$, $\epsilon \in (0, 1)$, $\ell \in [k-1]$, and $k \geq 3$, there exists no polynomial time approximation algorithm with an approximation ratio*

$$\frac{k}{2 + \frac{2(k-1)(k-2)\epsilon}{k+(k-2)\epsilon}} - \delta$$

*for the Vertex Cover problem in $\ell$-wise $\epsilon$-dense $k$-partite $k$-uniform hypergraphs assuming the Unique Games Conjecture.*

*Proof.* First, we concentrate on the $\epsilon$-dense case and afterwards, we extend the range of $\ell$. As a starting point of the reduction, we use the $k$-partite $k$-uniform hypergraph $H = (V_1, .., V_k, E(H))$ from Theorem 2 and construct an $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H' = (V_1', .., V_k', E')$.
Let us start with the description of $H'$. First, we join the set $C_i$ of $\frac{\epsilon}{1-\epsilon}\frac{n}{k}$ vertices to $V_i$ for every $i \in [k]$ and add all possible edges $e$ of $H'$ to $E'$ with the restriction $C_1 \cap e \neq \emptyset$. Thus, we obtain $|V_i'| = \frac{n}{k} + \frac{\epsilon}{1-\epsilon}\frac{n}{k}$ for all $i \in [k]$.
Now, let us analyze how the size of the optimal solution of $H'$ transforms. We denote by $OPT'$ an optimal vertex cover of $H'$. The UGC-hard decision question from Theorem 2 transforms into the following:

$$n\left(\frac{1}{2(k-1)} - \delta\right) + \frac{\epsilon}{1-\epsilon}\frac{n}{k} \leq |OPT'| \text{ or } |OPT'| \leq n\left(\frac{1}{k(k-1)} + \delta\right) + \frac{\epsilon}{1-\epsilon}\frac{n}{k}$$

Assuming the UGC, this implies the hardness of approximating the Vertex Cover problem in $\epsilon$-dense hypergraphs for every $\delta' > 0$ to within:

$$\frac{n\left(\frac{1}{2(k-1)} - \delta\right) + \frac{\epsilon}{1-\epsilon}\frac{n}{k}}{n\left(\frac{1}{k(k-1)} + \delta\right) + \frac{\epsilon}{1-\epsilon}\frac{n}{k}} = \frac{\frac{1-\epsilon}{2(k-1)} - \delta(1-\epsilon) + \frac{\epsilon}{k}}{\frac{1-\epsilon}{k(k-1)} + \delta(1-\epsilon) + \frac{\epsilon}{k}} \tag{15}$$

$$= \frac{\frac{(1-\epsilon)k}{2(k-1)k} + \frac{2\epsilon(k-1)}{2k(k-1)}}{\frac{1-\epsilon}{(k-1)k} + \frac{\epsilon(k-1)}{k(k-1)}} - \delta' \tag{16}$$

$$\frac{\frac{(1-\epsilon)k}{2(k-1)k}+\frac{2\epsilon(k-1)}{2k(k-1)}}{\frac{1-\epsilon}{(k-1)k}+\frac{\epsilon(k-1)}{k(k-1)}}-\delta' = \frac{\frac{k-\epsilon k+2\epsilon k-2\epsilon}{2(k-1)k}}{\frac{1-\epsilon+\epsilon k-\epsilon}{(k-1)k}}-\delta' \tag{17}$$

$$= \frac{k+(k-2)\epsilon}{2(1+(k-2)\epsilon)}-\delta' \tag{18}$$

$$= \frac{k}{\frac{2k(1+(k-2)\epsilon)}{k+(k-2)\epsilon}}-\delta' \tag{19}$$

$$= \frac{k}{\frac{2k+2(k-2)\epsilon+(2k-2)(k-2)\epsilon}{k+(k-2)\epsilon}}-\delta' \tag{20}$$

$$= \frac{k}{2+\frac{(2k-2)(k-2)\epsilon}{k+(k-2)\epsilon}}-\delta' \tag{21}$$

$$= \frac{k}{2+\frac{2(k-1)(k-2)\epsilon}{k+(k-2)\epsilon}}-\delta' \tag{22}$$

Finally, we have to verify that the constructed hypergraph $H'$ is indeed $\epsilon$-dense. Notice that $H'$ can have at most $(|V_1'|)^k = (\frac{n}{k}+\frac{\epsilon}{1-\epsilon}\frac{n}{k})^k$ edges. Therefore, we obtain the following:

$$\frac{\left(\frac{\epsilon}{1-\epsilon}\frac{n}{k}\right)\left(\frac{n}{k}+\frac{\epsilon}{1-\epsilon}\frac{n}{k}\right)^{k-1}}{\left(\frac{n}{k}+\frac{\epsilon}{1-\epsilon}\frac{n}{k}\right)^k} = \frac{\frac{n}{k}\frac{\epsilon}{1-\epsilon}}{\frac{n}{k}\left(1+\frac{\epsilon}{1-\epsilon}\right)} = \frac{\frac{\epsilon}{1-\epsilon}}{\frac{1+\epsilon-\epsilon}{1-\epsilon}} = \epsilon$$

Notice that the constructed hypergraph is also $\ell$-wise $\epsilon$-dense balanced. Hence, we obtain the same inapproximability factor in this case as well. ☐

Next, we combine the former construction with a conjecture about Unique Games hardness of the Vertex Cover problem in balanced $k$-partite $k$-uniform hypergraphs. In particular, we postulate the following:

**Conjecture 1.** *Given a balanced $k$-partite $k$-uniform hypergraph $H = (V_1, .., V_k, E(H))$ with $k \geq 3$, let $OPT$ denote an optimal vertex cover of $H$. For every $\delta > 0$, the following is UGC-hard to decide:*

$$|V|\left(\frac{1}{k}-\delta\right) \leq |OPT| \quad \text{or} \quad |OPT| \leq |V|\left(\frac{2}{k^2}+\delta\right)$$

Combining Conjecture 1 with the construction in Theorem 3, it yields the following inapproximability result which matches precisely the approximation upper bound achieved by our approximation algorithm described in Section 4:

**Theorem 4.** *For every $\delta > 0$, $\epsilon \in (0,1)$, $\ell \in [k-1]$, and $k \geq 3$, there exists no polynomial time approximation algorithm with an approximation ratio*

$$\frac{k}{2+(k-2)\epsilon}-\delta$$

*for the Vertex Cover problem in $\ell$-wise $\epsilon$-dense $k$-partite $k$-uniform hypergraphs assuming Conjecture 1.*

*Proof.* The UGC-hard decision question from Conjecture 1 transforms into the following:

$$n\left(\frac{1}{k}-\delta\right)+\frac{\epsilon}{1-\epsilon}\frac{n}{k} \leq |OPT'| \quad \text{or} \quad |OPT'| \leq n\left(\frac{2}{k^2}+\delta\right)+\frac{\epsilon}{1-\epsilon}\frac{n}{k}$$

Assuming the UGC, this implies the hardness of approximating the Vertex Cover problem in $\epsilon$-dense $k$-partite $k$-uniform hypergraphs for every $\delta' > 0$ to within:

$$\frac{n\left(\frac{1}{k}-\delta\right)+\frac{\epsilon}{1-\epsilon}\frac{n}{k}}{n\left(\frac{2}{k^2}+\delta\right)+\frac{\epsilon}{1-\epsilon}\frac{n}{k}} = \frac{n\left(\frac{1}{k}-\delta\right)(1-\epsilon)+\frac{\epsilon n}{k}}{n\left(\frac{2}{k^2}+\delta\right)(1-\epsilon)+\frac{\epsilon n}{k}} \quad (23)$$

$$= \frac{\frac{n}{k}}{n\left(\frac{2}{k^2}\right)(1-\epsilon)+\frac{k\epsilon n}{k^2}} - \delta' \quad (24)$$

$$= \frac{k}{2(1-\epsilon)+k\epsilon} - \delta' \quad (25)$$

$$= \frac{k}{2+(k-2)\epsilon} - \delta' \quad (26)$$

□

## 5.2 NP-Hardness

Recently, Sachdeva and Saket proved in [21] a nearly optimal NP-hardness of the Vertex Cover problem on balanced $k$-uniform $k$-partite hypergraphs. More precisely, they obtained the following inapproximability result:

**Theorem 5.** *[21] Given a balanced $k$-partite $k$-uniform hypergraph $H = (V, E)$ with $k \geq 4$, let $OPT$ denote an optimal vertex cover of $H$. For every $\delta > 0$, the following is NP-hard to decide:*

$$|V|\left(\frac{k}{2(k+1)(2(k+1)+1)}-\delta\right) \leq |OPT|$$

*or*

$$|V|\left(\frac{1}{k(2(k+1)+1)}+\delta\right) \geq |OPT|$$

Combining our reduction from Theorem 2 with Theorem 5, we prove the following inapproximability result under the assumption $P \neq NP$:

**Theorem 6.** *For every $\delta > 0$, $\epsilon \in (0, 1)$, $\ell \in [k-1]$, and $k \geq 4$, there is no polynomial time approximation algorithm with an approximation ratio*

$$\frac{k^2(1-\epsilon)+\epsilon 2(k+1)(2(k+1)+1)}{2(k+1)[1-\epsilon+\epsilon(2(k+1)+1)]} - \delta$$

*for the Vertex Cover problem in $\ell$-wise $\epsilon$-dense $k$-partite $k$-uniform hypergraphs assuming $P \neq NP$.*

*Proof.* The NP-hard decision question from Theorem 5 transforms into the following:

$$n\left(\frac{k}{2(k+1)(2(k+1)+1)} - \delta\right) + \frac{\epsilon}{1-\epsilon}\frac{n}{k} \leq |OPT'|$$

or

$$n\left(\frac{1}{k(2(k+1)+1)} + \delta\right) + \frac{\epsilon}{1-\epsilon}\frac{n}{k} \geq |OPT'|$$

Assuming $NP \neq P$, this implies the hardness of approximating the Vertex Cover problem in $\epsilon$-dense hypergraphs for every $\delta' > 0$ to within:

$$\frac{\frac{k(1-\epsilon)}{2(k+1)(2(k+1)+1)} + \frac{\epsilon}{k}}{\frac{1-\epsilon}{k(2(k+1)+1)} + \frac{\epsilon}{k}} - \delta' = \frac{\frac{k^2(1-\epsilon)+\epsilon 2(k+1)(2(k+1)+1)}{k2(k+1)(2(k+1)+1)}}{\frac{1-\epsilon+\epsilon(2(k+1)+1)}{k(2(k+1)+1)}} - \delta' \tag{27}$$

$$= \frac{k^2(1-\epsilon)+\epsilon 2(k+1)(2(k+1)+1)}{2(k+1)[1-\epsilon+\epsilon(2(k+1)+1)]} - \delta' \tag{28}$$

□

# 6 Further Research

An interesting question remains about even tighter lower approximation bounds for our problem, perhaps connecting it more closely to the integrality gap issue of the LP of Lovász [20].

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